

Orthogonal Matrix Polynomials: Zeros and Blumenthal's Theorem*

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Communicated by Walter Van Assche

Received June 24, 1994; accepted in revised form February 6, 1995

In this paper, we establish a quadrature formula and some basic properties of the zeros of a sequence $(P_n)_n$ of orthogonal matrix polynomials on the real line with respect to a positive definite matrix of measures. Using these results, we show how to get an orthogonalizing matrix of measures for a sequence $(P_n)_n$ satisfying a matrix three-term recurrence relation. We prove Blumenthal's theorem for orthogonal matrix polynomials describing the support of the orthogonalizing matrix of measures in case the matrix recurrence coefficients associated with these matrix polynomials tend to matrix limits having the same entries on every diagonal.

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1. INTRODUCTION

A close relationship between orthogonal matrix polynomials (or matrix polynomials satisfying a matrix three-term recurrence formula) and scalar polynomials satisfying a higher order recurrence formula has been established very recently (see [D1, D2, and DV]). Thus, in [DV], the following theorem has been proved:

THEOREM A. *Suppose $p_n(x)$ ($n = 0, 1, 2, \dots$) is a sequence of polynomials satisfying the following $(2N + 1)$ -term recurrence relation*

$$t^N p_n(t) = c_{n,0} p_n(t) + \sum_{k=1}^N (\overline{c_{n,k}} p_{n-k}(t) + c_{n+k,k} p_{n+k}(t)), \quad (1.1)$$

* This work has been partially supported by DGICYT Grant PB93-0926.

where $c_{n,0}$ ($n=0, 1, 2, \dots$) is a real sequence and $c_{n,k}$ ($n=0, 1, 2, \dots$) are complex sequences for $k=1, \dots, N$, with $c_{n,N} \neq 0$ for every n and with the initial conditions $p_k(x) = 0$ for $k < 0$ and p_k given polynomials of degree k , for $k=0, \dots, N-1$. We define the sequence of matrix polynomials $(P_n)_n$ by

$$P_n(t) = \begin{pmatrix} R_{N,0}(p_{nN})(t) & \cdots & R_{N,N-1}(p_{nN})(t) \\ R_{N,0}(p_{nN+1})(t) & \cdots & R_{N,N-1}(p_{nN+1})(t) \\ \vdots & \ddots & \vdots \\ R_{N,0}(p_{nN+N-1})(t) & \cdots & R_{N,N-1}(p_{nN+N-1})(t) \end{pmatrix}, \quad (1.2)$$

where the operator $R_{N,m}$ ($m=0, \dots, N-1$) is defined by

$$R_{N,m}(p)(t) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} t^n,$$

so that

$$p(t) = R_{N,0}(p)(t^N) + tR_{N,1}(t^N) + \cdots + t^{N-1}R_{N,N-1}(t^N).$$

Then the sequence of matrix polynomials defined by (1.2) is orthonormal on the real line with respect to a positive definite matrix of measures and satisfies a matrix three-term recurrence relation.

Conversely, suppose $P_n = (P_{n,m,j})_{m,j=0}^{N-1}$ is a sequence of orthonormal matrix polynomials or equivalently satisfying a matrix three-term recurrence relation (without loss of generality we can assume the leading coefficient of P_n to be a lower triangular matrix), then the scalar polynomials defined by

$$P_{nN+m}(t) = \sum_{j=0}^{N-1} t^j P_{n,m,j}(t^N), \quad (n \in \mathbb{N}, 0 \leq m \leq N-1),$$

satisfy a $(2N+1)$ -term recurrence relation of the form (1.1).

In this paper, we consider matrix polynomials satisfying the three-term recurrence relation

$$tP_n(t) = D_{n+1}P_{n+1}(t) + E_nP_n(t) + D_n^*P_{n-1}(t) \quad (1.3)$$

with $P_0(t) = I$ and $P_{-1}(t) = 0$, where $P_n(t)$ are matrix polynomials with coefficients in $\mathbb{C}^{N \times N}$ and the recurrence coefficients D_{n+1}, E_n are also $N \times N$ matrices. We can assume (see the proof of the previous theorem in [DV]) the matrices D_n to be lower triangular matrices with $\det D_n \neq 0$ and

$E_n^* = E_n$. Thus the expression of the matrices D_n, E_n , in terms of the recurrence coefficients which appear in (1.1), is

$$D_n = \begin{pmatrix} c_{nN, n} & 0 & 0 & \cdots & 0 \\ c_{nN, N-1} & c_{nN+1, N} & 0 & \cdots & 0 \\ c_{nN, N-2} & c_{nN+1, N-1} & c_{nN+2, N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{nN, 1} & c_{nN+1, 2} & c_{nN+2, 3} & \cdots & c_{nN+N-1, N} \end{pmatrix},$$

and

$$E_n = \begin{pmatrix} c_{nN, 0} & c_{nN+1, 1} & c_{nN+2, 2} & \cdots & c_{nN+N-1, N-1} \\ \overline{c_{nN+1, 1}} & \overline{c_{nN+1, 0}} & \overline{c_{nN+2, 1}} & \cdots & \overline{c_{nN+N-1, N-2}} \\ \overline{c_{nN+2, 2}} & \overline{c_{nN+2, 1}} & \overline{c_{nN+2, 0}} & \cdots & \overline{c_{nN+N-1, N-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{c_{nN+N-1, N-1}} & \overline{c_{nN+N-1, N-2}} & \overline{c_{nN+N-1, N-3}} & \cdots & \overline{c_{nN+N-1, 0}} \end{pmatrix}.$$

Matrix polynomials $(P_k(t))_k$ satisfying a recurrence formula of the form (1.3) are orthonormal with respect to some Hermitian matrix of measures $M = (\mu_{k,l})_{k,l=1}^N$ which is positive definite:

$$\int P_n(t) dM(t) P_m^*(t) = \delta_{m,n} I.$$

In Section 2 of this paper we study the properties of the zeros of the matrix polynomials $(P_n)_n$ (as usual, the zeros of the matrix polynomial P_n are the zeros of $\det P_n$), and establish a quadrature formula for the matrix inner product defined by the matrix of measures M . More precisely, we prove that

THEOREM 1.1. (a) *The zeros of P_n have a multiplicity not bigger than N . Furthermore P_n has nN zeros (taking into account their multiplicities) and all the zeros are real ($n \in \mathbb{N}$).*

(2) *If a is a zero of multiplicity p of P_n , then $\text{rank}(P_n(a)) = N - p$. If a is a zero of P_n and P_{n+1} , then $P_n(a)$ and $P_{n+1}(a)$ do not have any common eigenvector associated to 0.*

(3) *If we write $x_{n,k}$ ($k = 1, \dots, nN$) for the zeros of P_n ordered in increasing size (and taking into account their multiplicities), then*

$$x_{n+1, k} \leq x_{n, k} \leq x_{n+1, k+N} \quad \text{for } k = 1, \dots, nN.$$

(4) *If $x_{n,k}$ is a zero of P_n of multiplicity just N , then $P_n(a) = \theta$ (here and in the rest of this paper, we write θ for the null matrix, the dimension*

of which can be determined from the context. For instance, here θ is the $N \times N$ null matrix). Furthermore every complex value of $x_{n,k}^{1/N}$ is a zero of the N consecutive scalar polynomials $p_{nN}(t), \dots, p_{nN+N-1}(t)$. In this case the real number $x_{n,k}$ can not be a zero of the matrix polynomial P_{n+1} .

(5) Associated with every zero $x_{n,k}$ of the matrix polynomial P_n ($k = 1, \dots, nN$) there exists a $N \times N$ positive semidefinite matrix $B_{n,k}$ such that

$$\int P(t) dM(t) Q^*(t) = \sum_{k=1}^{nN} P(x_{n,k}) B_{n,k} Q^*(x_{n,k}) \tag{1.4}$$

for P, Q matrix polynomials satisfying $dgr(P) + dgr(Q) \leq 2n - 1$.

By using the relationship given in Theorem A and by considering the zeros of P_n as the eigenvalues of a certain Hermitian matrix, the proof of those results will be surprisingly simple. Using a different approach, other quadrature formulas have been found recently for orthonormal matrix polynomials by Sinap and Van Assche (see [SV]). The proof given here for the formula (1.4) should be compared with the proof given there. In a subsequent paper, one of the authors has completed Theorem 1.1 by giving new properties on the zeros of orthogonal matrix polynomials and a closed expression for the quadrature weights. These results have been used to extend Markov's theorem for orthogonal matrix polynomials (see [D3]).

From the $N \times N$ positive definite matrices $(B_{n,k})_{n,k}$ which appear in the quadrature formula (1.4), we can define a sequence of discrete positive definite matrices of measures by

$$\mu_n = \sum_{k=1}^{nN} B_{n,k} \delta_{x_{n,k}},$$

where $x_{n,k}$ ($k = 1, \dots, nN$) are the zeros of P_n . We complete Section 2 by proving that, as in the scalar case, an orthogonalizing measure for the matrix polynomials $(P_n)_n$ can be obtained as a limit point of these matrices of measures. Taking into account that this result will follow from Theorem 1.1 and that this theorem is a consequence of the fact that the matrix polynomials $(P_n)_n$ satisfy the matrix three-term recurrence relation (1.3), the results proved in this Section 2 provide a new proof of Favard's theorem for matrix polynomials satisfying a matrix three-term recurrence relation (for other proofs using different approaches, see [AN], or [D2]).

Finally, we establish in Section 3 Blumenthal's theorem for matrix polynomials. Indeed, we assume the recurrence coefficients in the formula (1.1) to be convergent sequences, i.e., for $k = 0, \dots, N$, the sequence $(c_{n,k})_n$ satisfies $c_k = \lim_{n \rightarrow \infty} c_{n,k}$. According to the relationship given in Theorem A, this is

equivalent to assuming that the matrix recurrence coefficients in (1.3) are converging to the matrices D, E which have equal entries on every diagonal (finite Toeplitz matrices), D is lower triangular and the entries of the upper i th diagonal of E are the same as the entries of the lower $(N+2-i)$ th diagonal of D . With this hypothesis, we prove that the support of the orthogonalizing matrix of measures (i.e., the support of the trace measure of this matrix of measures) is a compact interval and, possibly, two sequences of real numbers outside this interval which tend to the endpoints. This interval is given by

$$[c_0 + \inf_{x \in [-\pi, \pi]} t(x), c_0 + \sup_{x \in [-\pi, \pi]} t(x)],$$

where $t(x)$ is the trigonometric polynomial

$$t(x) = \left(\sum_{k=1}^N 2\Re(c_k) T_k(\cos x) \right) - \left(\sum_{k=1}^N 2 \sin(x) \Im(c_k) U_{k-1}(\cos x) \right),$$

and $(T_k)_k, (U_k)_k$ are the Chebyshev polynomials of the first and second kind, respectively. We give an example to show that the convergence of the matrix recurrence coefficients in (1.3) is not enough to guarantee the support of the matrix of measures to be a compact interval and, possibly, two sequences outside this interval which tend to the limit points of it. The Toeplitz nature of the limit matrices is needed for this structure of the support.

2. ZEROS OF ORTHOGONAL MATRIX POLYNOMIALS

In this Section, we study the zeros of a sequence $(P_n)_n$ of $N \times N$ matrix polynomials satisfying a three-term recurrence relation as (1.3); let us recall that we can assume the matrices D_k to be lower triangular matrices with $\det D_k \neq 0$ and $E_k^* = E_k$. As usual, a point x_0 is a zero of a matrix polynomial $P(x)$, if $\det P(x_0) = 0$, i.e., x_0 is a zero of the scalar polynomial $\det P(x)$.

Let us consider the infinite dimensional matrix defined by putting the sequences of matrices $(D_k)_k, (E_k)_k, (D_k^*)_k$, which appear in the recurrence formula (1.3), on the diagonals of the matrix J :

$$J = \begin{pmatrix} E_0 & D_1 & & & \\ D_1^* & E_1 & D_2 & & \\ & D_2^* & E_2 & D_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (2.1)$$

It is clear that J is a $(2N + 1)$ -banded infinite Hermitian matrix. We call this matrix the N -Jacobi matrix associated with the polynomials $(P_n)_n$. It is worth to note that this N -Jacobi matrix can be obtained by putting on the diagonals the recurrence sequences which appear in the $(2N + 1)$ -term recurrence formula (see (1.1)) satisfied by the scalar polynomials $(p_n)_n$, associated with $(P_n)_n$ according to Theorem A.

The N -Jacobi matrix is going to play a fundamental role in the study of the properties of the zeros of the matrix polynomials $(P_n)_n$, moreover, Theorem 1.1 will be a consequence of the following lemma.

LEMMA 2.1. *For $n \in \mathbb{N}$, the zeros of the matrix polynomial $P_n(t)$ are the same as those of the polynomial $\det(tI_{nN} - J_{nN})$ (with the same multiplicity), where I_{nN} is the identity matrix of dimension nN and J_{nN} is the truncated N -Jacobi matrix of dimension nN .*

To prove Lemma 2.1, we need the following lemma which is interesting in its own right. Note that for a given matrix A , we denote by $\text{Adj}(A)$ the classical adjoint, i.e., the matrix uniquely defined by the property

$$A \text{Adj}(A) = \text{Adj}(A) A = \det(A) I.$$

LEMMA 2.2. *Let $A(t)$ be a $N \times N$ matrix polynomial and let a be a zero of $A(t)$. We put*

$$R(a, A) = \{v \in \mathbb{C}^N : A(a) v^* = \theta\}.$$

If $\dim(R(a, A)) = p$, then

$$(\text{Adj}(A(t)))^{(l)}(a) = \theta, \quad \text{for } l = 0, \dots, p - 2,$$

and a is a zero of $A(t)$ of multiplicity at least p .

Proof of Lemma 2.2. Let us introduce some notation. We write $A_{i,j}(t)$ for the $(N - 1) \times (N - 1)$ matrix polynomial obtained from $A(t)$ by deleting its i th row and j th column. We write $r_{i,j}(t) = \det(A_{i,j}(t))$, i.e., the minor of the entry (i, j) of the matrix polynomial $A(t)$. Up to a sign the polynomial $r_{i,j}$ is the entry (j, i) of the matrix polynomial $\text{Adj}(A(t))$.

We must prove that $r_{i,j}^{(l)}(a) = 0$ for $l = 0, \dots, p - 2$.

We write $A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(t)$ for the matrix polynomial obtained by differentiating k_d ($k_d \geq 1$) times the column m_d ($d = 1, \dots, n \leq N$) of $A_{i,j}$. We then get the following expression for $r_{i,j}^{(l)}(a)$:

$$r_{i,j}^{(l)}(a) = \sum_{\sum_{d=1}^n k_d = l} \alpha_{(m_1, k_1), \dots, (m_n, k_n)} \det(A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(a)),$$

for certain nonnegative integers $\alpha_{(m_1, k_1), \dots, (m_n, k_n)}$. The result

$$(\text{Adj}(A(t)))^{(l)}(a) = \theta, \quad \text{for } l = 0, \dots, p-2$$

follows if we prove that for $0 \leq l \leq p-2$, $\sum_{d=1}^n k_d = l$ implies

$$\det(A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(a)) = 0.$$

Let us consider the subspace U of $R(a, A_{i,j})$ defined by

$$u \in U \quad \text{if and only if} \quad u_{m_1} = \dots = u_{m_n} = 0,$$

where u_m denotes the m th component of the vector u . From the definition of $A_{i,j}$, it is clear that

$$\dim(R(a, A_{i,j})) \geq \dim(R(a, A)) - 1.$$

We have then that

$$\begin{aligned} \dim(U) &\geq \dim(R(a, A_{i,j})) - n \geq \dim(R(a, A)) - n - 1 = p - n - 1 \\ &= p - 1 - \sum_{d=1}^n 1 \geq p - 1 - \sum_{d=1}^n k_d = p - 1 - l \geq 1. \end{aligned}$$

Hence, we can take $u \in U$, $u \neq \theta$.

The matrix $A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(a)$ differs from $A_{i,j}(a)$ just in the columns m_1, \dots, m_n . The vector u has just these components equal to 0, and since $u \in R(a, A_{i,j})$ it follows that

$$A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(a) u^* = A_{i,j}(a) u^* = \theta.$$

Since $u \neq \theta$, we have that

$$\det(A_{i,j, (m_1, k_1), \dots, (m_n, k_n)}(a)) = 0.$$

By differentiating the formula $\text{Adj}(A(t)) A(t) = \det A(t) I$ and taking into account what we have already proved, we obtain that

$$(\text{Adj}(A(t)))^{(l)}(a) A(a) = (\det A(t))^{(l)}(a) I, \quad \text{for } l = 0, \dots, p-1.$$

Since $A(a)$ is singular, we deduce that $(\det A(t))^{(l)}(a) = 0$, $l = 0, \dots, p-1$, and so a is a zero of $A(t)$ of multiplicity at least p . ■

We now prove Lemma 2.1

Proof of Lemma 2.1. Let a be an eigenvalue of the matrix J_{nN} , and let v an eigenvector ($v \neq \theta$) of this matrix corresponding to the eigenvalue a . We write v as a block column:

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}, \tag{2.2}$$

where $v_i \in \mathbb{C}^N$. The equation $J_{nN}v = av$ can be written

$$\begin{aligned} E_0 v_0 + D_1 v_1 &= av_0, \\ D_1^* v_0 + E_1 v_1 + D_2 v_2 &= av_1, \\ &\vdots \\ D_{n-1}^* v_{n-2} + E_{n-1} v_{n-1} &= av_{n-1}, \end{aligned}$$

which by the three-term recurrence relation for P_n , and using that D_k is non-singular, successively gives

$$\begin{aligned} v_1 &= P_1(a) v_0, \\ v_2 &= P_2(a) v_0, \\ &\vdots \\ v_{n-1} &= P_{n-1}(a) v_0, \\ \theta &= P_n(a) v_0. \end{aligned}$$

This shows that $v_0 \neq \theta$ (otherwise $v = \theta$), and hence $P_n(a)$ is singular, i.e., a is a zero of P_n .

If V_a is the space of right eigenvectors of the matrix J_{nN} for the eigenvalue a , then $W_a = \{v_0: v \in V_a\}$ is a subspace of \mathbb{C}^N of the same dimension as V_a , and $P_n(a) v_0 = \theta$, for $v_0 \in W_a$. This shows that $W_a \subset R(a, P_n)$, where by $R(a, P_n)$ we denote the space of right eigenvectors of the matrix $P_n(a)$ for the eigenvalue 0. Conversely, if $P_n(a) v_0 = \theta$ for some $v_0 \in \mathbb{C}^N$, and if we define $v_k = P_k(a) v_0$, $k = 1, \dots, n-1$, then v defined by (2.2) is an eigenvector for J_{nN} corresponding to a . This shows that $W_a = R(a, P_n)$. And so, we conclude that the dimension of $R(a, P_n)$ is just the multiplicity of a as an eigenvalue of J_{nN} . From Lemma 2.2, we deduce that a is a zero of $P_n(t)$ of multiplicity at least the multiplicity of a as an eigenvalue of J_{nN} .

The zeros of P_n are the zeros of $\det P_n$. Since the matrices $(D_k)_k$ are lower triangular and non-singular, from the matrix three-term recurrence formula it follows that $\det P_n$ is a polynomial of degree just nN . The matrix

J_{nN} is a $nN \times nN$ matrix, so $\det(tI_{nN} - J_{nN})$ is a polynomial of degree just nN , so from the result proved above, it follows that the zeros of the matrix polynomial P_n coincide with (and have the same multiplicity as) those of the polynomial $\det(tI_{nN} - J_{nN})$. Moreover, the matrix

$$A_a = \begin{pmatrix} P_0(a) \\ P_1(a) \\ \vdots \\ P_{n-1}(a) \end{pmatrix} \quad (2.3)$$

defines a bijection between $R(a, P_n)$ and the subspace of eigenvectors of the matrix J_{nN} associated with the eigenvalue a . ■

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. From the Lemma 2.1, and taking into account that the matrix J_{nN} is a $nN \times nN$ Hermitian $(2N + 1)$ -banded matrix, (1) of Theorem 1.1 follows. We have proved, in Lemma 2.1, that the multiplicity of a coincides with the dimension of $R(a, P_n)$, the space of right eigenvectors of the matrix $P_n(a)$ for the eigenvalue 0. But the dimension of this space is just $N - \text{rank}(P_n(a))$. If $P_n(a)$ and $P_{n+1}(a)$ had a common eigenvector v , then the recurrence formula would imply that a would be a zero of P_{n-1} , and then v would also be an eigenvector of $P_{n-1}(a)$ associated to 0. Proceeding successively we would obtain that a would be a zero of P_0 , which is a contradiction since P_0 is non-singular.

To prove (3) of Theorem 1.1, it will be enough to observe that the matrix J_{nN} is obtained from $J_{(n+1)N}$ by deleting the last N rows and columns, so the inclusion principle [HJ, p. 189] gives the separation properties for the zeros of P_n .

Now, we prove (4) of Theorem 1.1. If $x_{n,k}$ is a zero of P_n of multiplicity N , we deduce from part (2) of this theorem that $P_n(x_{n,k}) = \theta$, and then each vector u of \mathbb{C}^N is an eigenvector of $P_n(x_{n,k})$ associated to the eigenvalue 0. If $P_{n+1}(x_{n,k})$ was singular then $P_n(x_{n,k})$ and $P_{n+1}(x_{n,k})$ would have a common eigenvector associated to 0, which contradicts (2). The rest of (4) follows from the relationship between the orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence formula, which was given in the Introduction of this paper (see Theorem A).

Finally, we prove (5), i.e., the quadrature formula. Indeed, let us consider a zero $x_{n,l+1}$ of P_n with multiplicity just $m \leq N$. We can assume that $x_{n,l+1} = x_{n,l+2} = \dots = x_{n,l+m}$. We proved that the matrix $A_{x_{n,l+1}}$ (see (2.3)) establishes a bijection between the subspace $R(x_{n,l+1}, P_n)$ and the subspace of eigenvectors of the matrix J_{nN} associated with the eigenvalue $x_{n,l+1}$. Now, we choose a basis $\{v_{l+1}, \dots, v_{l+m}\}$ in $R(x_{n,l+1}, P_n)$ such that the vectors $A_{x_{n,l+1}} v_{l+1}, \dots, A_{x_{n,l+1}} v_{l+m}$ form an orthonormal basis of the

subspace of eigenvectors associated with the eigenvalue $x_{n,l+1}$ of the matrix J_{nN} . By proceeding in this way for every zero of the matrix polynomial P_n , and since eigenvectors corresponding to different eigenvalues are orthogonal, we obtain an orthonormal basis

$$\{A_{x_{n,1}}v_1, \dots, A_{x_{n,nN}}v_{nN}\}$$

in \mathbb{C}^{nN} . If we use these vectors as the columns of a matrix B , it is straightforward that this $nN \times nN$ matrix is unitary. The quadrature formula for the polynomials $(P_n)_n$ is implicitly involved in this property of the matrix B , as will be shown next.

According to the definition of the matrices $A_{x_{n,l}}$ (see (2.3)), we can write the matrix B as

$$B = (P_k(x_{n,l})v_l)_{\substack{k=0, \dots, n-1, \\ l=1, \dots, nN}}, \tag{2.4}$$

where $P_k(x_{n,l})v_l$ is a block of dimension $N \times 1$. Now, we consider the $nN^2 \times nN$ matrix C defined by

$$C = \begin{pmatrix} v_1 & \theta & \theta & \dots & \theta \\ \theta & v_2 & \theta & \dots & \theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \theta & \dots & v_{nN} \end{pmatrix}.$$

If we split up the $nN^2 \times nN^2$ matrix CC^* in blocks of dimension $N \times N$, we can write it as a block diagonal matrix

$$CC^* = \begin{pmatrix} B_{n,1} & & \\ & \ddots & \\ & & B_{n,nN} \end{pmatrix},$$

where the matrices $B_{n,k}$ are defined by $B_{n,k}(i,j) = v_{k,i}\bar{v}_{k,j}$, $i, j = 1, \dots, N$. So, these matrices are positive semidefinite. According to the definition of the matrix B (see (2.4)) and C , the condition $BB^* = I$ can be written as

$$\begin{pmatrix} P_0(x_{n,1}) & \dots & P_0(x_{n,nN}) \\ \vdots & \ddots & \vdots \\ P_{n-1}(x_{n,1}) & \dots & P_{n-1}(x_{n,nN}) \end{pmatrix} CC^* \begin{pmatrix} P_0^*(x_{n,1}) & \dots & P_{n-1}^*(x_{n,1}) \\ \vdots & \ddots & \vdots \\ P_0^*(x_{n,nN}) & \dots & P_{n-1}^*(x_{n,nN}) \end{pmatrix} = I.$$

If we split up the matrix

$$\begin{pmatrix} P_0(x_{n,1}) & \cdots & P_0(x_{n,nN}) \\ \vdots & \ddots & \vdots \\ P_{n-1}(x_{n,1}) & \cdots & P_{n-1}(x_{n,nN}) \end{pmatrix},$$

in the above product, in blocks of dimension $N \times nN$, according to the definition of the matrices $B_{n,k}$ ($k=1, \dots, nN$), we have

$$\sum_{i=1}^{nN} P_k(x_{n,i}) B_{n,i} P_l^*(x_{n,i}) = \delta_{k,l} I_{N \times N} \quad (2.5)$$

for $0 \leq k, l \leq n-1$. By definition of the vectors $(v_i)_{i=1, \dots, nN}$, we have that $P_n(x_{n,i}) v_i = v_i^* P_n^*(x_{n,i}) = \theta$, $i=1, \dots, nN$. This gives

$$(P_n(x_{n,1}), \dots, P_n(x_{n,nN})) C = \theta \quad \text{and} \quad C^* \begin{pmatrix} P_n^*(x_{n,1}) \\ \vdots \\ P_n^*(x_{n,nN}) \end{pmatrix} = \theta.$$

So, for $k, l=0, \dots, n-1$ we have that

$$\begin{cases} (P_k(x_{n,1}), \dots, P_k(x_{n,nN})) C C^* \begin{pmatrix} P_n^*(x_{n,1}) \\ \vdots \\ P_n^*(x_{n,nN}) \end{pmatrix} = \theta, \\ (P_n(x_{n,1}), \dots, P_n(x_{n,nN})) C C^* \begin{pmatrix} P_k^*(x_{n,1}) \\ \vdots \\ P_k^*(x_{n,nN}) \end{pmatrix} = \theta. \end{cases} \quad (2.6)$$

From (2.5) and (2.6), we get

$$\sum_{i=1}^{nN} P_k(x_{n,i}) B_{n,i} P_l^*(x_{n,i}) = \delta_{k,l} I_{N \times N}$$

for $k=0, \dots, n-1$ and $l=0, \dots, n$, or $k=0, \dots, n$ and $l=0, \dots, n-1$. The orthonormality of the polynomials $(P_n)_n$ gives that

$$\sum_{i=1}^{nN} P_k(x_{n,i}) B_{n,i} P_l^*(x_{n,i}) = \int P_k(t) dM(t) P_l^*(t)$$

for $k=0, \dots, n-1$ and $l=0, \dots, n$, or $k=0, \dots, n$ and $l=0, \dots, n-1$. That is, the quadrature formula (1.4) for the polynomials $(P_k)_{k=0}^{n-1}$, $(P_l)_{l=0}^n$ or $(P_k)_{k=0}^n$, $(P_l)_{l=0}^{n-1}$. By linearity, the quadrature formula will hold for all matrix polynomials P, Q for which $\text{dgr}(P) + \text{dgr}(Q) \leq 2n-1$. ■

Remark 2.3. Proceeding as in the proof of Lemma 2.1, it is possible to show that the zeros of certain perturbations of the matrix polynomial P_n are also real. Let A be a $N \times N$ matrix. If $D_n A = A^* D_n^*$, then the zeros of $P_n - AP_{n-1}$ are also real. Moreover, if a is a zero of multiplicity p of $P_n - AP_{n-1}$, then $\text{rank}(P_n(a) - AP_{n-1}(a)) = N - p$.

Proof. Let a be a zero of the matrix polynomial $P_n - AP_{n-1}$, and let v_0 be an eigenvector of the numerical matrix $P_n(a) - AP_{n-1}(a)$ associated to 0. Let us define $v_k = P_k(a) v_0$, $k = 1, \dots, n - 1$, and v as in (2.2). Then, using the three term recurrence formula (1.3) it is easy to show that v is an eigenvector of the matrix

$$\begin{pmatrix} E_0 & D_1 & & & & & & & & \\ D_1^* & E_1 & D_2 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & & D_{n-2}^* & E_{n-2} & D_{n-1} & & & & \\ & & & & D_{n-1}^* & E_{n-1} + D_n A & & & & \end{pmatrix}$$

associated to a . Since $D_n A$ is hermitian a must be real. The rest can be proved as in Lemma 2.1. ■

To complete this section, we show that an orthogonalizing positive definite matrix of measures for the polynomials $(P_n)_n$ can be obtained from the positive semidefinite matrices which appear in the quadrature formula (1.4). To do that, we consider the following positive definite and discrete matrices of measures

$$\mu_n = \sum_{k=1}^{nN} B_{n,k} \delta_{x_{n,k}}, \quad n \geq 0,$$

where $x_{n,k}$, $k = 1, \dots, nN$, are the zeros of the polynomial P_n (in increasing order and taking into account their multiplicities), and $B_{n,k}$, $k = 1, \dots, nN$ are the weights in the quadrature formula (1.4). We prove that, as in the scalar case, we can obtain an orthogonalizing positive definite matrix of measures for the polynomials $(P_n)_n$ as a limit point of these matrices of measures.

First of all, we need to recall some known results. By $\mathcal{C}([-a, b], \mathbb{C}^{N \times N})$ we denote the space of continuous functions from the interval $[-a, b]$ to the linear space of complex $N \times N$ matrices. The dual space $\mathcal{C}'([-a, b], \mathbb{C}^{N \times N})$ will be the space of $N \times N$ matrices whose entries are Borel measures. It is clear that the unit ball of $\mathcal{C}'([-a, b], \mathbb{C}^{N \times N})$ is weakly compact (Banach-Alaoglu).

Now, for $a, b \in \mathbb{R}^+$, we consider the matrix of measures

$$\mu_{n, [-a, b]} = \sum_{\substack{k=1 \\ x_{n,k} \in [-a, b]}}^{nN} B_{n,k} \delta_{x_{n,k}}.$$

Given two increasing sequences $(a_k)_k, (b_k)_k$, for which $a_k, b_k \rightarrow +\infty$ as $k \rightarrow +\infty$, doing a diagonal process and taking into account the weak compactness of the unit ball of the space $\mathcal{C}'([-a, b], \mathbb{C}^{N \times N})$, we obtain an increasing sequence of nonnegative integers $(n_m)_m$, and for every $k \in \mathbb{N}$ a matrix of measures $\mu^{(k)} \in \mathcal{C}'([-a_k, b_k], \mathbb{C}^{N \times N})$ such that

$$\lim_{m \rightarrow \infty} \int_{[-a_k, b_k]} f(t) d\mu_{n_m}(t) = \int_{[-a_k, b_k]} f(t) d\mu^{(k)}(t), \quad (2.7)$$

for all $f \in \mathcal{C}'([-a_k, b_k], \mathbb{C}^{N \times N})$. We can also get $\mu^{(k)} = \mu^{(k')}$ on $[-a_k, b_k]$ for $k \leq k'$. From these matrices of measures, we obtain a matrix of measures $\mu \in \mathcal{C}'(\mathbb{R}, \mathbb{C}^{N \times N})$ which extends the measures $\mu^{(k)}$, ($k \in \mathbb{N}$), i.e., the measure μ is such that for every $k \in \mathbb{N}$, $\mu = \mu^{(k)}$ in $[-a_k, b_k]$. Since the matrices of measures $(\mu_{n_m})_m$ are positive definite, it follows that the matrices of measures $\mu^{(k)}$ are positive definite, and so is μ .

Now, we prove that

$$\int_{\mathbb{R}} t^n I d\mu(t) = \lim_{m \rightarrow \infty} \int_{\mathbb{R}} t^n I d\mu_{n_m}(t), \quad \text{for all } n \in \mathbb{N}, \quad (2.8)$$

where I is the identity matrix of dimension N . For fixed $n \in \mathbb{N}$ the quadrature formula (1.4) gives for $n_m, n_{m'} > n$ that

$$\int_{\mathbb{R}} t^n I d\mu_{n_m}(t) = \int_{\mathbb{R}} t^n I d\mu_{n_{m'}}(t).$$

If we write A_n for this matrix, we must prove that $A_n = \int_{\mathbb{R}} t^n I d\mu(t)$. Let $\|\cdot\|_2$ be the spectral norm defined by

$$\|A\|_2 = \max \{ \sqrt{|\lambda|} : \lambda \text{ is an eigenvalue of } A^*A \}.$$

For $n_m > n$, we have

$$\begin{aligned} & \left\| \int_{-a_k}^{b_k} t^n I d\mu(t) - A_n \right\|_2 \\ &= \left\| \int_{-a_k}^{b_k} t^n I d\mu(t) - \int_{\mathbb{R}} t^n I d\mu_{n_m}(t) \right\|_2 \\ &\leq \left\| \int_{-a_k}^{b_k} t^n I d\mu(t) - \int_{-a_k}^{b_k} t^n I d\mu_{n_m}(t) \right\|_2 \\ &\quad + \left\| \int_{-\infty}^{-a_k} t^n I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^n I d\mu_{n_m}(t) \right\|_2. \end{aligned}$$

Given $\varepsilon > 0$, since $\mu = \mu^{(k)}$ in $[-a_k, b_k]$, for n_m big enough, (2.7) gives

$$\left\| \int_{-a_k}^{b_k} t^n I d\mu(t) - \int_{-a_k}^{b_k} t^n I d\mu_{n_m}(t) \right\|_2 < \varepsilon.$$

Now, if we take a nonnegative integer l such that $2l > n$, we have that

$$|t^n| = \left| \frac{t^{2l}}{t^{2l-n}} \right| \leq \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} t^{2l} \quad \text{for } t \notin [-a_k, b_k].$$

Hence, we have the following matrix inequalities (as usual $A \leq B$ if $B - A$ is positive semidefinite)

$$\begin{aligned} & - \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \left(\int_{-\infty}^{-a_k} t^{2l} I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^{2l} I d\mu_{n_m}(t) \right) \\ & \leq \int_{-\infty}^{-a_k} t^n I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^n I d\mu_{n_m}(t) \\ & \leq \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \left(\int_{-\infty}^{-a_k} t^{2l} I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^{2l} I d\mu_{n_m}(t) \right). \end{aligned}$$

Since $t^{2l} \geq 0$, for $t \in \mathbb{R}$, and the matrix of measures μ_{n_m} is positive definite, it follows that the matrix

$$\left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \left(\int_{-\infty}^{-a_k} t^{2l} I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^{2l} I d\mu_{n_m}(t) \right)$$

is also positive definite. By the definition of the spectral norm $\|\cdot\|_2$, we get

$$\begin{aligned} & \left\| \int_{-\infty}^{-a_k} t^n I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^n I d\mu_{n_m}(t) \right\|_2 \\ & \leq \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \left\| \int_{-\infty}^{-a_k} t^{2l} I d\mu_{n_m}(t) + \int_{b_k}^{\infty} t^{2l} I d\mu_{n_m}(t) \right\|_2 \\ & \leq \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \left\| \int_{\mathbb{R}} t^{2l} d\mu_{n_m}(t) \right\|_2 \\ & = \left(\frac{1}{\min \{a_k, b_k\}} \right)^{2l-n} \|A_{2l}\|_2 \end{aligned}$$

Let $k \rightarrow \infty$, then this proves (2.8).

From (2.8), and the quadrature formula (1.4), it follows that the polynomials $(P_n)_n$ are orthonormal with respect to the positive definite matrix of

measures μ . Observe that μ may not be unique since we have selected only one possible weak limit of the sequence of discrete measures given by the quadrature formula.

3. BLUMENTHAL'S THEOREM FOR ORTHOGONAL MATRIX POLYNOMIALS

In this section we extend Blumenthal's theorem regarding the support of the orthogonalizing measure for a sequence of orthogonal polynomials on the real line in case the recurrence coefficients associated with these polynomials tend to finite limits, i.e., the orthogonalizing measure μ belongs to the Nevai class $M(a, b)$, for certain $a > 0$ and $b \in \mathbb{R}$.

Our extension is to orthogonal matrix polynomials for which the recurrence matrix coefficients satisfy a similar condition.

We consider again a sequence $(P_n)_n$ of $N \times N$ matrix polynomials satisfying a three-term recurrence relation of the form (1.3). Without loss of generality, we can assume the matrices D_k to be lower triangular matrices with $\det D_k \neq 0$ and $E_k^* = E_k$. We also consider the scalar polynomials $(p_n)_n$ associated with the matrix polynomials $(P_n)_n$ as in theorem A of the introduction. The polynomials satisfy a $(2N+1)$ -term recurrence relation of the form (1.1). Then, we can assume that the recurrence coefficients in this recurrence relation tend to finite limits. We denote these limits by c_0, \dots, c_N :

$$c_k = \lim_{n \rightarrow \infty} c_{n,k}, \quad k = 0, \dots, N.$$

Since the sequence $(c_{n,0})_n$ is real, the number c_0 is also real.

According to the relationship, given in the introduction of this paper, between the matrix coefficients in the recurrence formula for $(P_n)_n$ and the scalar coefficients in the $(2N+1)$ -term recurrence formula for $(p_n)_n$, the hypothesis is equivalent to the following: the matrix coefficients $(D_k)_k$, $(E_k)_k$ tend to the matrix limits $D = (D_{i,j})_{i,j=1, \dots, N}$, $E = (E_{i,j})_{i,j=1, \dots, N}$, which satisfy

$$\begin{aligned} \text{for } i < j, \quad D_{i,j} &= 0, \\ \text{for } m = 0, \dots, N-1, \quad D_{i+m,i} &= D_{j+m,j}, \quad i, j = 1, \dots, N-m, \\ \text{for } m = 0, \dots, N-1, \quad E_{i+m,i} &= E_{j+m,j}, \quad i, j = 1, \dots, N-m, \\ \text{and for } m = 2, \dots, N, \quad E_{1,m} &= D_{N-m+2,1} \end{aligned}$$

that is, the matrix limit D is lower triangular, D and E have equal entries on every diagonal (finite Toeplitz matrices) and the entries of the upper i th

diagonal of E are the same as the entries of the lower $N + 2 - i$ th diagonal of D . Define the support of μ by

$$\text{supp}(\mu) = \text{supp}(\text{tr}(\mu)) = \text{supp}(\mu_{1,1} + \mu_{2,2} + \cdots + \mu_{N,N}). \tag{3.1}$$

We can now prove the following theorem

THEOREM 3.1. *Assume that the coefficients in the matrix three-term recurrence relation converge to Toeplitz matrices. Define the trigonometric polynomial*

$$t(x) = \left(\sum_{k=1}^N 2\Re(c_k) T_k(\cos x) \right) - \left(\sum_{k=1}^N 2 \sin(x) \Im(c_k) U_{k-1}(\cos x) \right),$$

where $(T_k)_k$ and $(U_k)_k$ are the Chebyshev polynomials of the first and second kind, respectively. Let μ be the positive definite matrix of measures with respect to which the matrix polynomials $(P_n)_n$ are orthonormal. Then the support of μ is the compact interval $[c_0 + \inf_{x \in [-\pi, \pi]} t(x), c_0 + \sup_{x \in [-\pi, \pi]} t(x)]$ and, possibly, two sequences of real numbers outside this interval which tend to the endpoints. More precisely

$$\left[c_0 + \inf_{x \in [-\pi, \pi]} t(x), c_0 + \sup_{x \in [-\pi, \pi]} t(x) \right] \subset \text{supp}(\mu),$$

and for every $\varepsilon > 0$ the set

$$\text{supp}(\mu) \setminus [c_0 + \inf_{x \in [-\pi, \pi]} t(x) - \varepsilon, c_0 + \sup_{x \in [-\pi, \pi]} t(x) + \varepsilon]$$

is finite.

Before proving the theorem, we give an example proving that the theorem is not true (the support of μ does not need to be a compact interval and, possibly two sequences tending to the endpoints) if we assume that the matrix recurrence coefficients $(D_k)_k, (E_k)_k$ tend to matrix limits D, E without the restriction of equal entries on the diagonal. Let μ_1 and μ_2 be two positive measures in the classes $M(a, b)$ and $M(a', b')$, respectively, where $a \neq a'$, and $b \neq b'$. Let us put $(p_{n,1})_n, (p_{n,2})_n$ for the orthonormal polynomials associated with μ_1, μ_2 , respectively. Define the positive definite matrix of measures ν by

$$\nu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

If we write the recurrence relations for $(p_{n,1})_n, (p_{n,2})_n$ in the form

$$\begin{aligned} a_{n+1,1} p_{n+1,1}(t) + b_{n,1} p_{n,1}(t) + a_{n,1} p_{n-1,1}(t) &= t p_{n,1}(t), \\ a_{n+1,2} p_{n+1,2}(t) + b_{n,2} p_{n,2}(t) + a_{n,2} p_{n-1,2}(t) &= t p_{n,2}(t), \end{aligned}$$

then it is not hard to see that the matrix recurrence coefficients for the orthonormal matrix polynomials with respect to ν are

$$D_k = \begin{pmatrix} a_{k,1} & 0 \\ 0 & a_{k,2} \end{pmatrix}, \quad E_k = \begin{pmatrix} b_{k,1} & 0 \\ 0 & b_{k,2} \end{pmatrix}.$$

These recurrence coefficients tend to the matrices

$$D = \begin{pmatrix} \frac{a}{2} & 0 \\ 0 & a' \end{pmatrix}, \quad E = \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}.$$

But the support of ν is $[b-a, b+a] \cup [b'-a', b'+a']$ and, possibly, some sequences tending to the endpoints of these intervals. Taking $b+a < b'-a'$, the support of ν is not an interval.

Proof of Theorem 3.1. We consider the $(2N+1)$ -banded Jacobi matrix associated with the matrix polynomials $(P_n)_n$ (see (2.1)). The sequences on the diagonals of this infinite matrix, that is, $(c_{n+k,k})_n$, $k=0, \dots, N$ and $(\overline{c_{n,k}})_n$, $k=1, \dots, N$, are (by hypothesis) convergent sequences, with limits $c_k = \lim_{n \rightarrow \infty} c_{n,k}$, $k=0, \dots, N$. This implies that the operator associated with the matrix J in the Hilbert space l^2 and defined by

$$J: l^2 \rightarrow l^2$$

$$J((a_n)_n) = (a_n)_n J,$$

is bounded. In [D2, Sect. 3], we show how to get an orthogonalizing matrix of measures for $(P_n)_n$ from the resolution of the identity of any self-adjoint extension of the operator J . From the results proved there and the boundedness of the operator J , it follows that the orthogonalizing matrix of measures of the matrix polynomials $(P_n)_n$ is unique, and that its support (defined by (3.1)) coincides with the spectrum of J . So, we are going to determine this spectrum.

We proceed as in [MNV, Sect. 4]. First, by using the following theorem of H. Weyl, we replace the matrix J by a simpler one.

THEOREM (Weyl). *Let A and B be bounded self-adjoint operators on a Hilbert space, and assume that B is compact. Then the essential spectra of A and $A+B$ are the same.*

The essential spectrum of an operator is defined as the set of limit points of its spectrum. According to this theorem, we can replace the matrix J by the $(2N+1)$ -banded infinite matrix J_0 having the entries on every diagonal equal to the limit of the corresponding diagonal in the matrix J . Indeed,

since the diagonals of the $(2N + 1)$ -banded infinite matrix $J - J_0$ tend to zero, the operator defined by this matrix is compact, and so, J and J_0 have the same essential spectrum. Theorem 3.1 will follow if we prove that the spectrum of J_0 is the compact interval $[c_0 + \inf_{x \in [-\pi, \pi]} t(x), c_0 + \sup_{x \in [-\pi, \pi]} t(x)]$.

We can assume that $c_N \neq 0$. Indeed, if not, we would have to determine the spectrum of an operator defined by a $(2M + 1)$ -banded infinite matrix ($M < N$), instead of a $(2N + 1)$ -banded infinite matrix. Consider the operator J_0 acting on the Hardy space H^2 , that is, the Hilbert space of analytic functions f on the unit disk D

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

with Taylor coefficients $(a_j)_j$ belonging to l^2 , equipped with the norm

$$\|f\| = \left(\sum_{j=0}^{\infty} |a_j|^2 \right)^{1/2}.$$

Using the isomorphism

$$(a_j)_{j=0}^{\infty} \mapsto \sum_{j=0}^{\infty} a_j z^j$$

between l^2 and H^2 , the operator J can be represented on H^2 as

$$\begin{aligned} (J_0 f)(z) &= c_N z^N f(z) + \dots + c_1 z f(z) + c_0 f(z) + \overline{c_1} \frac{f(z)}{z} + \dots + \overline{c_N} \frac{f(z)}{z^N} \\ &\quad - f(0) \left(\frac{\overline{c_1}}{z} + \dots + \frac{\overline{c_N}}{z^N} \right) - f'(0) \left(\frac{\overline{c_2}}{z} + \dots + \frac{\overline{c_N}}{z^{N-1}} \right) \\ &\quad - \dots - \frac{f^{(N-1)}(0)}{(N-1)!} \frac{\overline{c_N}}{z}. \end{aligned}$$

A complex number λ belongs to the spectrum of J_0 , if the operator $J_0 - \lambda I$ does not have a bounded inverse (I is the identity operator) in H^2 . Let us consider the equation $(J_0 - \lambda I)f = g$, where $g \in H^2$ is given. This equation can be written as

$$\begin{aligned} &\left(c_N z^N + \dots + c_1 z + c_0 - \lambda + \frac{\overline{c_1}}{z} + \dots + \frac{\overline{c_N}}{z^N} \right) f(z) \\ &\quad - f(0) \left(\frac{\overline{c_1}}{z} + \dots + \frac{\overline{c_N}}{z^N} \right) - \dots - \frac{f^{(N-1)}(0)}{(N-1)!} \frac{\overline{c_N}}{z} = g(z). \end{aligned}$$

Solving the equation for $f(z)$ we get:

$$f(z) = \frac{z^N g(z) + f(0)(\overline{c_1} z^{N-1} + \dots + \overline{c_N}) + \dots + \frac{f^{(N-1)}(0)}{(N-1)!} \overline{c_N} z^{N-1}}{c_N z^{2N} + \dots + c_1 z^{N+1} + (c_0 - \lambda) z^N + \overline{c_1} z^{N-1} + \dots + \overline{c_N}}. \quad (3.2)$$

Let us write

$$p_\lambda(z) = c_N z^{2N} + \dots + c_1 z^{N+1} + (c_0 - \lambda) z^N + \overline{c_1} z^{N-1} + \dots + \overline{c_N}.$$

We have to determine when the equation (3.2) defines a bounded operator in H^2 . We prove that this precisely happens when exactly N of the roots of $p_\lambda(z)$ are inside the unit disk $D = \{z: |z| < 1\}$. Observe that the $2N$ roots of the polynomial $p_\lambda(z)$ are of the form $x_1, \dots, x_N, 1/\overline{x_1}, \dots, 1/\overline{x_N}$.

Suppose first that none of the roots $x_1, \dots, x_N, 1/\overline{x_1}, \dots, 1/\overline{x_N}$ has modulus one. Then N of these roots are inside D . We can suppose these to be x_1, \dots, x_N . The function f defined by (3.2) will be analytic on D if and only if x_1, \dots, x_N are also zeros of the numerator in (3.2). This gives a linear system of equations with the N unknowns $f(0), \dots, f^{(N-1)}(0)/(N-1)!$ and the number of equations is equal to the number of different roots in x_1, \dots, x_N . If all these roots are different we have a square linear system whose determinant is

$$\det \begin{pmatrix} \overline{c_1} x_1^{N-1} + \dots + \overline{c_N} & \overline{c_2} x_1^{N-1} + \dots + \overline{c_N} x_1 & \dots & \overline{c_N} x_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{c_1} x_N^{N-1} + \dots + \overline{c_N} & \overline{c_2} x_N^{N-1} + \dots + \overline{c_N} x_N & \dots & \overline{c_N} x_N^{N-1} \end{pmatrix}$$

which after some straightforward simplifications becomes

$$\overline{c_N}^N \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{N-1} \\ 1 & x_2 & \dots & x_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \dots & x_N^{N-1} \end{pmatrix},$$

which is a Vandermonde determinant with value $\overline{c_N}^N \prod_{1 \leq i < j \leq N} (x_j - x_i)$. If all the roots x_1, \dots, x_N are different, then for any given g in H^2 we can determine uniquely the corresponding f in H^2 . Then the operator defined by (3.2) is one-to-one and onto. Hence, it has a bounded inverse, according to the open mapping theorem.

Let's see now what happens if the multiplicity of any of the roots x_1, \dots, x_N is greater than 1. Suppose we have p different roots x_1, \dots, x_p , with multiplicities N_1, \dots, N_p , respectively, and $N_1 + \dots + N_p = N$. In this case, to compensate for the roots of $p_\lambda(z)$ we force the numerator in (3.2) to have a zero of order just N_m in x_m , ($m = 1, \dots, p$). Then we get again

a square linear system with the same unknowns as before. Taking into account that the derivative of order j of the numerator in (3.2) is

$$(z^N g(z))^{(j)} + f(0) \left(\frac{(N-1)!}{(N-j-1)!} \overline{c}_1 z^{N-j-1} + \dots + j! \overline{c}_{N-j} \right) + \dots + \frac{f^{(N-1)}(0)}{(N-1)!} \frac{(N-1)!}{(N-j-1)!} \overline{c}_N z^{N-j-1}, \quad 0 \leq j \leq N$$

we have that the determinant of this system is

$$\det \begin{pmatrix} \overline{c}_1 x_1^{N-1} + \dots + \overline{c}_N & \dots & \overline{c}_N x_1^{N-1} \\ (N-1) \overline{c}_1 x_1^{N-2} + \dots + \overline{c}_{N-1} & \dots & (N-1) \overline{c}_N x_1^{N-2} \\ \vdots & \vdots & \vdots \\ \frac{(N-1)!}{(N-N_1)!} \overline{c}_1 x_1^{N-N_1} + \dots + (N_1-1)! \overline{c}_{N-N_1+1} & \dots & \frac{(N-1)!}{(N-N_1)!} \overline{c}_N x_1^{N-N_1} \\ \vdots & \vdots & \vdots \\ \overline{c}_1 x_p^{N-1} + \dots + \overline{c}_N & \dots & \overline{c}_N x_p^{N-1} \\ (N-1) \overline{c}_1 x_p^{N-2} + \dots + \overline{c}_{N-1} & \dots & (N-1) \overline{c}_N x_p^{N-2} \\ \vdots & \vdots & \vdots \\ \frac{(N-1)!}{(N-N_p)!} \overline{c}_1 x_p^{N-N_p} + \dots + (N_p-1)! \overline{c}_{N-N_p+1} & \dots & \frac{(N-1)!}{(N-N_p)!} \overline{c}_N x_p^{N-N_p} \end{pmatrix}$$

which after some simplifications becomes

$$\overline{c}_N^N \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{N_1} & \dots & x_1^{N_p} & \dots & x_1^{N-1} \\ 0 & 1 & \dots & N_1 x_1^{N_1-1} & \dots & N_p x_1^{N_p-1} & \dots & (N-1) x_1^{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & N_1! & \dots & \frac{N_p!}{(N_p-N_1+1)!} x_1^{N_p-N_1+1} & \dots & \frac{(N-1)!}{(N-N_1)!} x_1^{N-N_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_p & \dots & x_p^{N_1} & \dots & x_p^{N_p} & \dots & x_p^{N-1} \\ 0 & 1 & \dots & N_1 x_p^{N_1-1} & \dots & N_p x_p^{N_p-1} & \dots & (N-1) x_p^{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & N_p! & \dots & \frac{(N-1)!}{(N-N_p)!} x_p^{N-N_p} \end{pmatrix}$$

which we can express in terms of partial derivatives of the Vandermonde determinant:

$$\begin{aligned} & \overline{c}_N^N \frac{\partial^{1+\dots+N_1-1+\dots+1+\dots+N_p-1}}{\partial y_2 \dots \partial y_{N_1}^{N_1-1} \dots \partial y_{N-N_p+1} \dots \partial y_N^{N_p-1}} \\ & \times \left(\det \begin{pmatrix} 1 & y_1 & \dots & y_1^{N-1} \\ 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_N & \dots & y_N^{N-1} \end{pmatrix} \right)_{\substack{y_1 = \dots = y_{N_1} = x_1 \\ \dots \\ y_{N-N_p+1} = \dots = y_N = x_p}} \\ & = \overline{c}_N^N \frac{\partial^{1+\dots+N_1-1+\dots+1+\dots+N_p-1}}{\partial y_2 \dots \partial y_{N_1}^{N_1-1} \dots \partial y_{N-N_p+1} \dots \partial y_N^{N_p-1}} \\ & \times \left(\prod_{1 \leq i < j \leq N} (y_j - y_i) \right)_{\substack{y_1 = \dots = y_{N_1} = x_1 \\ \dots \\ y_{N-N_p+1} = \dots = y_N = x_p}} \end{aligned}$$

which is not zero. Again, for any given g in H^2 , we can determine uniquely the corresponding f in H^2 . Thus the operator defined by (3.2) is one-to-one and onto. Hence, it has a bounded inverse, according to the open mapping theorem.

So, we have proved that when exactly N of the roots of $p_\lambda(z)$ are inside the unit disk D , the equation (3.2) defines a bounded operator in H^2 . On the other hand, if λ is such that some of the roots x_1, \dots, x_N of the polynomial p_λ have modulus 1, then we must compensate for the roots of the denominator in D but also for those on the unit circle T , in order that the function f defined by (3.2) belongs to H^2 . But in this case, we get a linear system of equations with N unknowns and more than N equations (let us recall that the roots of p_λ are $x_1, \dots, x_N, 1/\overline{x_1}, \dots, 1/\overline{x_N}$). The coefficient matrix for this system is as before (but not a square matrix), and so the rank of this matrix is N . However, it is very easy to see that an appropriate choice of the function g gives an augmented matrix for this system with rank just $N+1$, and so, for this function g , the equation (3.2) does not have a solution in H^2 . For this λ , the operator $J_0 - \lambda I$ does not have a bounded inverse in H^2 .

Hence, we have proved that for a given λ the equation (3.2) defines a bounded operator in H^2 if and only if the polynomial

$$p_\lambda(z) = c_N z^{2N} + \dots + c_1 z^{N+1} + (c_0 - \lambda) z^N + \overline{c}_1 z^{N-1} + \dots + \overline{c}_N$$

has exactly N roots inside the unit disk $D = \{z: |z| < 1\}$. So the spectrum of J_0 are the λ 's for which $p_\lambda(z)$ has at least one root on $T = \{|z| = 1\}$. But saying this is the same as saying that the equation

$$c_0 - \lambda = -c_N z^N - \overline{c}_N \frac{1}{z^N} - \dots - c_1 z - \overline{c}_1 \frac{1}{z} \quad (3.3)$$

has a solution in T . If we write

$$h(z) = -c_N z^N - \overline{c_N} \frac{1}{z^N} - \dots - c_1 z - \overline{c_1} \frac{1}{z},$$

then the equation (3.3) has a solution in T if and only if

$$\inf_{z \in T} h(z) \leq c_0 - \lambda \leq \sup_{z \in T} h(z).$$

So we deduce that the spectrum of J_0 is the compact interval

$$I = [c_0 - \sup_{z \in T} h(z), c_0 - \inf_{z \in T} h(z)].$$

But according to the definition of Chebyshev polynomials of first and second kind, we conclude that for $z = e^{ix}$, $x \in [-\pi, \pi]$

$$\begin{aligned} h(z) &= -c_N z^N - \overline{c_N} \frac{1}{z^N} - \dots - c_1 z - \overline{c_1} \frac{1}{z} \\ &= -\left(\sum_{k=1}^N 2\Re(c_k) T_k(\cos x) \right) + \left(\sum_{k=1}^N 2 \sin(x) \Im(c_k) U_{k-1}(\cos x) \right) = -t(x). \end{aligned}$$

And the proof of Theorem 3.1 is finished.

ACKNOWLEDGMENT

We thank the referee for his/her useful comments and suggestions, especially on the Proof of Lemma 2.1.

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